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A Study of Curvature Tensors By Using Berwald's and Cartan's Higher-Order Derivatives in Finsler Spaces

Adel Mohammed Ali Al-Qashbari [1,2] & Fahmi Ahmed Mothana AL-ssallal [1]

[1] Department. of Math's., Faculty of Education. Aden University, Aden, Yemen
[2] Department of Engineering, Faculty of the Engineering and Computers, University of Science & Technology-Aden, Yemen

[*]*Corresponding author:* Adel_ma71@yahoo.com

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Abstract

This research investigates the decomposition of curvature tensors in Finsler spaces using higher-order derivatives of Berwald and Cartan connections. By employing these derivatives, we aim to provide a more comprehensive understanding of the geometric structure of Finsler spaces. Previous studies have explored various types of recurrences and their implications for curvature tensors. However, a systematic analysis of decomposition using higher-order derivatives has been lacking. This paper fills this gap by introducing a new approach to decompose curvature tensors and analyzing the properties of the resulting components. Our findings contribute to the existing body of knowledge on Finsler geometry and may have potential applications in related fields. In this paper, we investigate some identities between Weyl's curvature tensor and Cartan's 3rd Curvature Tensor R^i_{jkh} . We first introduce the basic concepts of Weyl's tensor W^i_{jkh} and Cartan's 3th Curvature Tensor R^i_{jkh}. Then, we derive some identities between these two tensors. Finally, we apply these identities to some examples.

Keywords: Covariant Derivative of second orders, Weyl Tensor W^i_{jkh} ,Cartan's 3th , Curvature Tensor R^i_{jkh} and Cartan's 4th Curvature Tensor K^i_{jkh}

الملخص

يُعنى هذا البحث بتحليل موترات الانحناء في الأوساط الفينرية باستخدام المشتقات العليا لاتصالات بروالد وكارتان . ويهدف هذا البحث، من خلال استخدام هذه المشتقات، إلى تقديم فهم أكثر شمولية للبنية الهندسية للأوساط الفينرية . وقد تناولت الدراسات السابقة أنواعاً مختلفة من التكرار وآثارها على موترات الانحناء، إلا أنه كان هناك نقص في التحليل المنهجي للتحليل باستخدام المشتقات العليا . ويملأ هذا البحث هذه الفجوة من خلال تقديم نهج جديد لتحليل

موترات الانحناء وتحليل خصائص المكونات الناتجة . تساهم نتائجنا في المعرفة الحالية حول الهندسة التفاضلية وقد يكون لها تطبيقات محتملة في المجالات ذات الصلة.

في هذه الورقة البحثية، نستطلع على بعض التطابقات بين مُوتر انحناء ويل (ويُعرف أيضًا بمُوتر انحناء التطابق) والموتر التقوسي الثالث لكارتان R^i_{jkh} أولاً، نقدم المفاهيم الأساسية لكل من مُوتر انحناء ويل وموتر التقوسي الثالث لكارتان R^i_{jkh} ، ثم نستنتج بعض التطابقات بين هذين المُوترين. وأخيرا، نطبق هذه التطابقات على بعض الأمثلة.

كلمات مفتاحية: مشتقة باروارد من الرتبة الثانية، للموتر الأسقاطي لويلي W^i_{jkh} ، وللموتر التقوسي الثالث لكارتان R^i_{jkh}.

INTRODUCTION

Finsler geometry, a generalization of Riemannian geometry, has garnered significant attention due to its applications in various fields, including physics and engineering. A fundamental aspect of Finsler geometry is the study of curvature tensors, which provide insights into the geometric properties of these spaces. In particular, the decomposition of curvature tensors has been a subject of interest for many researchers. Curvature tensors play a pivotal role in understanding the geometric properties of Finsler spaces. While previous research has focused on different types of recurrences and their implications for curvature tensors, a comprehensive study of decomposition using higher-order derivatives of Berwald and Cartan connections is still needed.

This paper addresses this gap by investigating the decomposition of curvature tensors in Finsler spaces using higher-order derivatives. By leveraging these derivatives, we aim to gain deeper insights into the structure and properties of curvature tensors. The results of this study are expected to contribute to the development of Finsler geometry and provide a foundation for further research in this area.

Previous studies, such as those by Al-Qashbari [5, 6, 7, 8, 9, 10], Qasem [16, 17, 18], and Abdallah [1] have explored different types of recurrences and their implications for curvature tensors in Finsler spaces. Additionally, researchers like Ahsan and Ali [3, 4] have investigated curvature tensors in the context of general relativity. However, a comprehensive analysis of curvature tensor decomposition using higher-order derivatives of Berwald and Cartan connections remains an open area.

This paper aims to contribute to this field by investigating the decomposition of curvature tensors in Finsler spaces utilizing the higher-order derivatives of Berwald and Cartan connections. By building upon the foundational work of previous researchers, we seek to uncover new insights into the structure and properties of these tensors. The results of this study are expected to have implications for various areas of Finsler geometry and its applications.

Weyl's tensor W^i_{jkh} and conformal tensor C^i_{jkh} are two important geometric objects in differential geometry. They are both used to study the curvature of spacetime. Weyl's tensor W^i_{jkh} is a conformal invariant, which means that it is invariant under conformal transformations. The conformal tensor C^i_{jkh} is not a conformal invariant, but it is related to Weyl's tensor W^i_{jkh} by a simple formula.

In this paper, we investigate some identities between Weyl's tensor W^i_{jkh} and conformal tensor C^i_{jkh} . We first introduce the basic concepts of Weyl's curvature tensor and conformal curvature tensor. Then, we derive some identities between these two tensors. Finally, we apply these identities to some examples.

The metric tensor g_{ij} and $\mathcal{B}_k$ (Berwald's connection coefficients) G^i_{jk} are positively homogeneous of degree 0 in directional arguments.

Two vectors y_i and y^i meet the following conditions

a) $y_i = g_{ij}\, y^j$, b) $y_i\, y^i = F^2$, c) $\delta^k_j y^j = y^k$

d) $g_{ir}\, \delta^i_j = g_{rj}$ and e) $g^{jk}\delta^i_k = g^{ji}$. (1.1)

The quantities g_{ij} and g^{ij} are related by [11]

$$g_{ij}\, g^{jk} = \delta^k_i = \begin{cases} 1 , & if \quad i = k , \\ 0 , & if \quad i \neq k . \end{cases} \tag{1.2}$$

Tensor C_{ijk} is known as (h)hv-torsion tensor defined by

$$C_{ijk} = \frac{1}{2}\dot{\partial}_i\, g_{jk} = \frac{1}{4}\dot{\partial}_i\dot{\partial}_j\dot{\partial}_k\, F^2. \tag{1.3}$$

The (v)hv-torsion tensor C^h_{ik} and tensor C_{ijk} are given by

a) $C^i_{jk}y^j = C^i_{jk}y^k = 0$, b) $C_{ijk}\, y^i = C_{ijk}\, y^j = C_{ijk}\, y^k = 0$,

c) $g^{jk}C_{ijk} = C_i$ and d) $g^{jk}C_{ijh} = C^k_{ih}$. (1.4)

Covariant derivative $\mathcal{B}_k T^i_j$ for Berwald's ($\mathcal{B}_k$) of any tensor T^i_j w. r. t. x^k is defined as

$$\mathcal{B}_k T^i_j = \partial_k T^i_j - \left(\dot{\partial}_r T^i_j\right) G^r_k + T^r_j\, G^i_{rk} - T^i_r G^r_{jk} . \tag{1.5}$$

The vector y^i and metric function F are vanished identically for Berwald's covariant derivative.

a) $\mathcal{B}_k F = 0$ and b) $\mathcal{B}_k y^i = 0$. (1.6)

Metric tensor g_{ij} is not equal to zero (i.e. not vanish), defined by

$$\mathcal{B}_k g_{ij} = -2\, C_{ijk|h}\, y^h = -2\, y^h \mathcal{B}_h C_{ijk} . \tag{1.7}$$

Tensor W^i_{jkh} , torsion tensor W^i_{jk} and deviation tensor W^i_j are defined by:

$$W^i_{jkh} = H^i_{jkh} + \frac{2\,\delta^i_j}{(n+1)} H_{[hk]} + \frac{2\,y^i}{(n+1)} \dot{\partial}_j H_{[kh]} + \frac{\delta^i_k}{(n^2-1)} \big(n\, H_{jh} + H_{hj} + y^r \dot{\partial}_j H_{hr}$$

$$- \frac{\delta^i_h}{(n^2-1)} \big(n\, H_{jk} + H_{kj} + y^r \dot{\partial}_j H_{kr} \big) \quad , \tag{1.8}$$

$$W^i_{jk} = H^i_{jk} + \frac{y^i}{(n+1)} H_{[jk]} + 2\,\{ \frac{\delta^i_{[j}}{(n^2-1)} \big(n\, H_{k]} - y^r H_{k]\,r} \big) \} \ . \tag{1.9}$$

and $W^i_j = H^i_j - H\,\delta^i_j - \frac{1}{(n+1)} \big(\dot{\partial}_r H^r_j - \dot{\partial}_j H \big)\, y^i$, respectively.

(1.10) The tensors W^i_{jkh} , W^i_{jk} and W_{jk} satisfy the following identities

a) $W^i_{jkh}\, y^j = W^i_{kh}$, b) $W^i_{kh}\, y^k = W^i_h$, c) $W^i_{jki} = W_{jk}$

d) $g_{ir}\, W^i_{jkh} = W_{rjkh}$ e) $W^i_{jkh} = - W^i_{jhk}$ and f) $W^i_{jkh} + W^i_{khj} + W^i_{hjk} = 0$. (1.11)

Also, if we suppose that the tensor W^i_j satisfy the following identities

a) $W^i_k\, y^k = 0$, b) $W^i_i = 0$, c) $W^i_k\, y_i = 0$,

d) $g_{ir}\, W^i_j = W_{rj}$, $e)\ g^{jk} W_{jk} = W$ and f) $W_{jk}\, y^k = 0$. (1.12)

The skew-symmetric in its indices k and h in the tensor W^i_{jkh} .

Cartan's 3[th] curvature tensor R^i_{jkh} , Ricci tensor R_{jk} , the vector H_k and scalar curvature H are defined as

a) $R^i_{jkh} = \Gamma^{*i}_{hjk} + \big(\Gamma^{*i}_{ljk}\big)\, G^l_h + C^i_{jm}\big(G^m_{kh} - G^m_{kl}\, G^l_h \big) + \Gamma^{*i}_{mk}\, \Gamma^{*m}_{jh} - k/h$,

b) $R^i_{jkh}\, y^j = H^i_{kh}$, c) $R_{jk}\, y^j = H_k$, d) $R_{jk}\, y^k = R_j$,

e) $R^i_i = R$, f) $g_{ir}\, R^i_{jkh} = R_{rjkh}$, g) $R^i_{jkh} = - R^i_{jhk}$,

h) $g^{jk} R^i_{jkh} = R^i_h$, i) $R^i_{jki} = R_{jk}$, t) $H_i\, y^i = H^i_i = (n-1)\, H$

j) $H^i_{kh}\, y^k = H^i_h$ and k) $H^i_{ki} = H_k$. (1.13)

Cartan's 4[th]curvature tensor K^i_{jkh}, Ricci tensor K_{jk},vector K_k and scalar curvature K are defined as

a) $K^i_{jkh}\, y^j = H^i_{kh}$, b) $K_{jk}\, y^j = H_k$, c) $K_{jk}\, y^k = K_j$, d) $g^{jk}\, K_{jk} = K$,

e) $K^i_{jki} = K_{jk}$ and f) $g^{jk}\, K^i_{jkh} = K^i_h$. (1.14)

We consider an n-dimensional Finsler space F_n in which the Weyls projective curvature tensor denoted by W^i_{jkh} satisfies the following condition (AL-Qashbari & AL-Maisary [10])

$$\mathcal{B}_m W^i_{jkh} = \lambda_m W^i_{jkh} + \mu_m \big(\delta^i_k\, g_{jh} - \delta^i_h g_{jk} \big) \, . \tag{1.15}$$

From (1.15) have studied the generalized this space, (1.15) which can be written as

$$\mathcal{B}_m W^i_{jkh} = \lambda_m W^i_{jkh} + \mu_m \big(\delta^i_k\, g_{jh} - \delta^i_h g_{jk} \big) + \frac{1}{4} \big(W^i_k\, g_{jh} - W^i_h g_{jk} \big) \, . \tag{1.16}$$

GENERALIZED -$\mathcal{B}$W-BIRECURRENT SPACE

In this paper, we introduce a new class of Finsler spaces, namely, generalized-BW-birecurrent spaces. These spaces generalize the concept of birecurrence to a broader setting and exhibit interesting geometric properties. We investigate the curvature tensor of these spaces and establish several characterization theorems. Our work in this paper we defined $\mathcal{B}_m\mathcal{B}_l$ is covariant derivative of second order.

Taking the covariant derivative of (1.16), with respect to x^l in the sence of Berwald, we get

$$\mathcal{B}_l\,\mathcal{B}_m W^i_{jkh} = (\mathcal{B}_l\,\lambda_m)\mathrm{W}^i_{jkh} + \lambda_m\big(\mathcal{B}_l\mathrm{W}^i_{jkh}\big) + (\mathcal{B}_l\mu_m)\big(\delta^i_k\, g_{jh} - \delta^i_h g_{jk}\big)$$
$$+\,\mu_m\mathcal{B}_l\big(\delta^i_k\, g_{jh} - \delta^i_h g_{jk}\big) + \tfrac{1}{4}\mathcal{B}_l\,\big(W^i_k\, g_{jh} - W^i_h g_{jk}\big)\,. \tag{2.1}$$

Using (1.7) and (1.15) in (2.1), we get

$$\mathcal{B}_l\mathcal{B}_m W^i_{jkh} = \lambda_{ml}\mathrm{W}^i_{jkh} + \lambda_m\left(\lambda_l\mathrm{W}^i_{jkh} + \mu_l\big(\delta^i_k\, g_{jh} - \delta^i_h g_{jk}\big) + \tfrac{1}{4}\,\big(W^i_k\, g_{jh} - W^i_h g_{jk}\big)\right)$$
$$+\,\mu_{ml}\big(\delta^i_k\, g_{jh} - \delta^i_h g_{jk}\big) - 2\mu_m\mathcal{B}_q y^q\big(\delta^i_k\, C_{jhl} - \delta^i_h C_{jkl}\big)$$
$$+\,\tfrac{1}{4}\,\big((\mathcal{B}_l W^i_k)\, g_{jh} - (\mathcal{B}_l W^i_h) g_{jk}\big) - \tfrac{1}{2}\,\mathcal{B}_q y^q\big(W^i_k\, C_{jhl} - W^i_h C_{jkl}\big)\,.$$

Or

$$\mathcal{B}_l\,\mathcal{B}_m W^i_{jkh} = (\lambda_{ml} + \lambda_m\lambda_l)\mathrm{W}^i_{jkh} + (\mu_{ml} + \lambda_m\mu_l)\big(\delta^i_k\, g_{jh} - \delta^i_h g_{jk}\big)$$
$$+\tfrac{1}{4}\big((\mathcal{B}_l W^i_k)\, g_{jh} - (\mathcal{B}_l W^i_h)\, g_{jk}\big) + \tfrac{1}{4}\,\lambda_m\big(W^i_k\, g_{jh} - W^i_h g_{jk}\big)$$
$$-2\,\mu_m\mathcal{B}_q y^q\big(\delta^i_k C_{jhl} - \delta^i_h C_{jkl}\big) - \tfrac{1}{2}\,\mathcal{B}_q y^q\big(W^i_k\, C_{jhl} - W^i_h C_{jkl}\big)\,. \tag{2.2}$$

The equation (2.2), can be written as

$$\mathcal{B}_l\,\mathcal{B}_m W^i_{jkh} = a_{ml}\mathrm{W}^i_{jkh} + b_{ml}\big(\delta^i_k\, g_{jh} - \delta^i_h g_{jk}\big) + \tfrac{1}{4}\big((\mathcal{B}_l W^i_k)\, g_{jh} - (\mathcal{B}_l W^i_h) g_{jk}\big)$$
$$+\,\tfrac{1}{4}\lambda_m\,\big(W^i_k\, g_{jh} - W^i_h g_{jk}\big) - 2\mu_m\mathcal{B}_q y^q\big(\delta^i_k\, C_{jhl} - \delta^i_h C_{jkl}\big)$$
$$-\,\tfrac{1}{2}\,\mathcal{B}_q y^q\big(W^i_k\, C_{jhl} - W^i_h C_{jkl}\big)\,. \tag{2.3}$$

where $a_{ml} = \lambda_{ml} + \lambda_m\lambda_l$ and $b_{ml} = \mu_{ml} + \lambda_m\mu_l$ are non-zero covariant tensors field of second order, respectively.

Definition 2. 1. A Finsler space of tensor W^i_{jkh} is called as Wely's projective curvature tensor and is known as satisfies the condition (2.3), will be called a generalized birecurrent space. We shall call this Finsler space as a generalized $\mathcal{B}W$-birecurrent space and we denoted by G$\mathcal{B}$W-BRF$_n$.

Result 2.1. All a generalized $\mathcal{B}W$-recurrent space is a generalized $\mathcal{B}W$ birecurrent space.

Transvecting condition to a higher dimensional space (2.3) by y^j, using (1.1a), (1.4b), (1.6b) and (1.11a), we get

$$\mathcal{B}_l\, \mathcal{B}_m W^i_{kh} = a_{ml} \mathrm{W}^i_{kh} + b_{ml}\big(\delta^i_k\, y_h\, - \delta^i_h y_k\,\big) + \tfrac{1}{4}\,\big((\mathcal{B}_l W^i_k)\, y_h\, - (\mathcal{B}_l W^i_h)\, y_k\,\big)$$

$$+\,\lambda_m\, \tfrac{1}{4}\,\big(W^i_k\, y_h\, - W^i_h y_k\,\big)\,. \tag{2.4}$$

Again, transvecting condition to a higher dimensional space (2.4) by y^k, using (1.1b), (1.2), (1.6b), (1.11b) and (1.12a), we get

$$\mathcal{B}_l\, \mathcal{B}_m W^i_h = a_{ml} \mathrm{W}^i_h + b_{ml}\big(y^i\, y_h\, - \delta^i_h\, F^2\big) - \tfrac{1}{4}(\mathcal{B}_l W^i_h)F^2 - \tfrac{1}{4}\lambda_m\, W^i_h F^2\,. \tag{2.5}$$

Therefore, the proof of theorem is completed, we can say

Theorem 2.1. In G$\mathcal{B}$W-BRF$_n$, covariant derivative for Berwald of second order for torsion tensor W^i_{kh} and deviation tensor W^i_h are given by (2.4) and (2.5).

Contracting the indices space by summing over i and h in the conditions (2.3) and using (1.1b), (1.2), (1.11c), (1.12b) and (1.12d), we get

$$\mathcal{B}_l\, \mathcal{B}_m W_{jk} = a_{ml} \mathrm{W}_{jk} + b_{ml}(1-n) g_{jk}\, + \tfrac{1}{4}\, \mathcal{B}_l\, \mathrm{W}_{jk} + \tfrac{1}{4}\, \lambda_m \mathrm{W}_{jk}$$

$$-2\mu_m \mathcal{B}_q y^q (1-n) C_{jkl} - \tfrac{1}{2}\, \mathcal{B}_q y^q W^i_k\, C_{jil}\,. \tag{2.6}$$

Thus, we conclude

Theorem 2.2. In G$\mathcal{B}$W-BRF$_n$, the Ricci W_{jk} is generalized birecurrent Finsler space given by the equation (2.6).

Transvecting (2.6) by g^{jk}, and using (1.1e), (1.4c), (1.4d) and (1.12e), we get

$$\mathcal{B}_l\, \mathcal{B}_m W\, = a_{ml} W + b_{ml}\,(1-n)\, + \tfrac{1}{4}\, \mathcal{B}_l\, W\, + \tfrac{1}{4}\, \lambda_m W\, - 2\mu_m \mathcal{B}_q y^q (1-n) C_l$$

$$-\tfrac{1}{2}\, \mathcal{B}_q y^q W^i_k\, C^k_{il}\,. \tag{2.7}$$

From conditions (2.7), we show that the curvature scalar W cannot equal to zero because if the vanishing of W would imply $a_{ml} = 0$ and $b_{ml} = 0$, that is a contradiction.

Thus, we conclude

Theorem 2.3. In G$\mathcal{B}$W-BRF$_n$, the scalar W in equations (2.7) is non-vanishing.

RELATIONSHIP BETWEEN WELY'S CURVATURE TENSOR AND CARTAN'S 3TH CURVATURE TENSOR R^i_{jkh}

Finsler geometry, as a generalization of Riemannian geometry, provides a powerful framework for modeling a wide range of physical phenomena. In Finsler spaces, the curvature properties of the space are characterized by various curvature tensors, among which Weyl and Cartan's third curvature tensors play a significant role. While the geometric interpretations and physical implications of these tensors have been extensively studied, the relationship between them remains a subject of ongoing research.

This paper aims to investigate the connection between Weyl's curvature tensor and Cartan's third curvature tensor in Finsler spaces. By exploring their algebraic and geometric properties, we seek to establish new identities and inequalities that relate these two tensors. Our findings are expected to contribute to a deeper understanding of the curvature structure of Finsler spaces and provide insights into their applications in physics, such as in the study of gravitational theories and cosmology.

Some properties of W^i_{jkh} curvature tensor was proposed by Ahsan and Ali [3],[4] in (2014).

For $(n = 4)$ a Riemannian space, it is known that Cartan's 3[th] curvature tensor R^i_{jkh} and Wely's projective curvature tensor W^i_{jkh} are connected by the formula [1]

$$W^i_{jkh} = R^i_{jkh} + \frac{1}{3}\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right) . \tag{3.1}$$

Taking the covariant derivative of (3.1), with respect to x^m and x^l in the sence of Berwald we get

$$\mathcal{B}_l \mathcal{B}_m W^i_{jkh} = \mathcal{B}_l \mathcal{B}_m R^i_{jkh} + \frac{1}{3} \mathcal{B}_l \mathcal{B}_m\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right) . \tag{3.2}$$

Using (1.7), in (3.2) we get

$$\mathcal{B}_l \mathcal{B}_m W^i_{jkh} = \mathcal{B}_l \mathcal{B}_m R^i_{jkh} + \frac{1}{3} \mathcal{B}_l \mathcal{B}_m \left(\delta^i_k R_{jh} - g_{jk} R^i_h\right) + \frac{2}{3} \mathcal{B}_q\, y^q\, C_{jkl}\, \mathcal{B}_m R^i_h$$
$$+ \frac{2}{3} \mathcal{B}_q\, y^q \mathcal{B}_l\left(C_{jkm} R^i_h\right) . \tag{3.3}$$

Using (2.3) and (3.2) in (3.3), we get

$$\mathcal{B}_l \mathcal{B}_m R^i_{jkh} = a_{ml} \mathrm{R}^i_{jkh} + b_{ml}\left(\delta^i_k\, g_{jh} - \delta^i_h g_{jk}\right) + \frac{1}{3} a_{ml}\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right)$$
$$+ \frac{1}{4}\left((\mathcal{B}_l W^i_k) g_{jh} - (\mathcal{B}_l W^i_h) g_{jk}\right) + \frac{1}{4} \lambda_m\left(W^i_k g_{jh} - W^i_h g_{jk}\right) - 2\mu_m \mathcal{B}_q y^q\left(\delta^i_k C_{jhl} - \delta^i_h C_{jkl}\right)$$
$$- \frac{1}{2} \mathcal{B}_q y^q\left(W^i_k C_{jhl} - W^i_h C_{jkl}\right) - \frac{1}{3} \mathcal{B}_l \mathcal{B}_m\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right) - \frac{2}{3} \mathcal{B}_q\, y^q\, C_{jkl}\, \mathcal{B}_m R^i_h$$
$$- \frac{2}{3} \mathcal{B}_q\, y^q \mathcal{B}_l\left(C_{jkm} R^i_h\right) . \tag{3.4}$$

This shows that

$$\mathcal{B}_l \mathcal{B}_m R^i_{jkh} = a_{ml} \mathrm{R}^i_{jkh} + b_{ml}\left(\delta^i_k\, g_{jh} - \delta^i_h g_{jk}\right) + \frac{1}{4}\left((\mathcal{B}_l W^i_k)\, g_{jh} - (\mathcal{B}_l W^i_h) g_{jk}\right)$$
$$+ \frac{1}{4} \lambda_m\left(W^i_k g_{jh} - W^i_h\, g_{jk}\right) - 2\mu_m \mathcal{B}_q y^q\left(\delta^i_k C_{jhl} - \delta^i_h C_{jkl}\right) - \frac{1}{2} \mathcal{B}_q y^q\left(W^i_k C_{jhl} - W^i_h C_{jkl}\right)$$
$$- \frac{2}{3} \mathcal{B}_q\, y^q\, C_{jkl}\, \mathcal{B}_m R^i_h - \frac{2}{3} \mathcal{B}_q\, y^q \mathcal{B}_l\left(C_{jkm} R^i_h\right) . \tag{3.5}$$

If and only if

$$\mathcal{B}_l \mathcal{B}_m\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right) = a_{ml}\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right) . \tag{3.6}$$

Thus, we conclude

Theorem 3.1. In G$\mathcal{B}$W-BRF$_n$, Cartan's 3th curvature tensor R^i_{jkh} is a generalized birecurrent Finsler space if and only if the tensor $\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right)$ is a generalized birecurrent Finsler space.

Transvecting condition (3.4) by y^j, using (1.6b), (1.1a), (1.4b), (1.13b) and (1.13c), we get

$$\mathcal{B}_l \mathcal{B}_m H^i_{kh} = a_{ml} H^i_{kh} + b_{ml}\left(\delta^i_k y_h - \delta^i_h y_k\right) + \frac{1}{3} a_{ml}\left(\delta^i_k H_h - y_k R^i_h\right)$$
$$+\frac{1}{4}\left((\mathcal{B}_l W^i_k) y_h - (\mathcal{B}_l W^i_h) y_k\right) + \frac{1}{4}\lambda_m\left(W^i_k y_h - W^i_h y_k\right)$$
$$-\frac{1}{3}\mathcal{B}_l \mathcal{B}_m\left(\delta^i_k H_h - y_k R^i_h\right) . \qquad (3.7)$$

This shows that

$$\mathcal{B}_l \mathcal{B}_m H^i_{kh} = a_{ml} H^i_{kh} + b_{ml}\left(\delta^i_k y_h - \delta^i_h y_k\right) . \qquad (3.8)$$

If and only if

$$\mathcal{B}_l \mathcal{B}_m\left(\delta^i_k H_h - y_k R^i_h\right) = a_{ml}\left(\delta^i_k H_h - y_k R^i_h\right) . \qquad (3.9)$$

and

$$(\mathcal{B}_l W^i_k) y_h - (\mathcal{B}_l W^i_h) y_k = \lambda_m\left(W^i_k y_h - W^i_h y_k\right) . \qquad (3.10)$$

The proof of theorem is completed, we conclude

Theorem 3.2. In G$\mathcal{B}$W-BRF$_n$, the covariant derivative of the second orders for the torsion tensor H^i_{kh} is a generalized birecurrent Finsler space if and only if (3.9) and (3.10) holds good.

Transvecting (3.7) by y^k, using (1.6b), (1.1b), (1.1c), (1.12a) and (1.13j), we get

$$\mathcal{B}_l \mathcal{B}_m H^i_h = a_{ml} \mathrm{H}^i_h + b_{ml}\left(y^i y_h - \delta^i_h F^2\right) - \frac{1}{4}(\mathcal{B}_l W^i_h) F^2 - \frac{1}{4}\lambda_m W^i_h F^2$$
$$+\frac{1}{3} a_{ml}\left(y^i H_h - F^2 R^i_h\right) - \frac{1}{3}\mathcal{B}_l \mathcal{B}_m\left(y^i H_h - F^2 R^i_h\right) . \qquad (3.11)$$

This shows that

$$\mathcal{B}_l \mathcal{B}_m H^i_h = a_{ml} \mathrm{H}^i_h + b_{ml}\left(y^i y_h - \delta^i_h F^2\right) . \qquad (3.12)$$

If and only if

$$\mathcal{B}_l \mathcal{B}_m\left(y^i H_h - F^2 R^i_h\right) = a_{ml}\left(y^i H_h - F^2 R^i_h\right) \qquad (3.13)$$

and

$$(\mathcal{B}_l W^i_h) F^2 = \lambda_m W^i_h F^2 . \qquad (3.14)$$

Thus, we conclude

Theorem 3.3. In G$\mathcal{B}$W-BRF$_n$, the covariant derivative of the second orders for the deviation tensor H^i_h is a generalized birecurrent Finsler space if and only if (3.13) and (3.14), holds good.

Contracting the indices i and h in the equations (3.7) and (3.11), respectively and using (1.2), (1.1a), (1.1b), (1.13k), (1.13t), (1.12c) and (1.12b), we get

$$\mathcal{B}_l\mathcal{B}_m H_k = a_{ml}H_k + b_{ml}(1-n)\, y_k + \frac{1}{3}a_{ml}(H_k - y_k R) - \frac{1}{3}\mathcal{B}_l\mathcal{B}_m(H_k - y_k\, R\,). \quad (3.15)$$

This shows that

$$\mathcal{B}_l\, \mathcal{B}_m H_k = a_{ml}H_k + \ b_{ml}(1-n)\, y_k\,. \quad (3.16)$$

If and only if

$$\mathcal{B}_l\, \mathcal{B}_m(H_k - \ y_k\, R\,) = a_{ml}(\, H_k - \ y_k\, R\,)\,. \quad (3.17)$$

And

$$\mathcal{B}_l\mathcal{B}_m H = a_{ml}H + b_{ml}(1-n)\, F^2 + \frac{1}{3}a_{ml}(H\ -\ F^2 R\,) - \frac{1}{3}\,\mathcal{B}_l\,\mathcal{B}_m(H -\ F^2 R\,). \quad (3.18)$$

This shows that

$$\mathcal{B}_l\, \mathcal{B}_m H\ = a_{ml}H + b_{ml}(1-n)\, F^2. \quad (3.19)$$

If and only if

$$\mathcal{B}_l\, \mathcal{B}_m(H\ - F^2 R\,) = a_{ml}(\, H - F^2 R\,)\,. \quad (3.20)$$

Thus, we conclude

Theorem 3.4. In G$\mathcal{B}$W-BRF$_n$, vector H_k and scalar H are given in (3.16) and (3.19) if and only if the conditions (3.17) and (3.20) are holds good, respectively.

Contracting the indices i and h in the equations (3.7) and using (1.1d), (1.1b), (1.13i), (1.13e), (1.12d) and (1.12b), we get

$$\mathcal{B}_l\, \mathcal{B}_m R_{jk} = a_{ml}R_{jk} + b_{ml}(1-n)\, g_{jk} + \frac{1}{4}\mathcal{B}_l \mathrm{W}_{jk} + \frac{1}{4}\lambda_m W_{jk} - 2\mu_m \mathcal{B}_q y^q\, (1-n)\mathrm{C}_{jkl}$$
$$- \frac{1}{2}\,\mathcal{B}_q y^q W_k^i\, \mathrm{C}_{jil} - \frac{1}{3}\,\mathcal{B}_l\,\mathcal{B}_m\left(\, R_{jk} - g_{jk}\, R\,\right) + \frac{1}{3}a_{ml}\left(\, R_{jk} - g_{jk}\, R\,\right)$$
$$- \frac{2}{3}\mathcal{B}_q\, y^q\, C_{jkl}\, \mathcal{B}_m R\ - \frac{2}{3}\mathcal{B}_q\, y^q \mathcal{B}_l\left(C_{jkm} R\,\right). \quad (3.21)$$

This shows that

$$\mathcal{B}_l\, \mathcal{B}_m R_{jk} = a_{ml}R_{jk} + b_{ml}(1-n)\, g_{jk} + \frac{1}{4}\mathcal{B}_l W_{jk} + \frac{1}{4}\lambda_m W_{jk} - 2\mu_m \mathcal{B}_q y^q (1-n)\, \mathrm{C}_{jkl}$$
$$- \frac{1}{2}\,\mathcal{B}_q y^q W_k^i\, \mathrm{C}_{jil} - \frac{2}{3}\mathcal{B}_q\, y^q\, C_{jkl}\, \mathcal{B}_m R\ - \frac{2}{3}\mathcal{B}_q\, y^q \mathcal{B}_l\left(C_{jkm} R\,\right). \quad (3.22)$$

If and only if

$$\mathcal{B}_l\, \mathcal{B}_m\left(\, R_{jk} - g_{jk}\, R\,\right) = a_{ml}\left(\, R_{jk} -\ g_{jk}\, R\,\right). \quad (3.23)$$

In conclusion the proof of theorem is completed, we get

Theorem 3.5. In G$\mathcal{B}$W-BRF$_n$, R-Ricci tensor R_{jk} is given in (3.22), if and only if the condition (3.23) is holds good.

Transvecting the equation (3.21) by y^k, using (1.6b), (1.1a), (1.1c), (1.4b), (1.12a), (1.12f) and (1.13d), we get

$$\mathcal{B}_l\,\mathcal{B}_m R_j = a_{ml} R_j + b_{ml}(1-n) y_j + \tfrac{1}{3} a_{ml}\big(R_j - y_j\,R\big) - \tfrac{1}{3}\mathcal{B}_l\,\mathcal{B}_m\big(R_j\, - y_j\,R\,\big)\,. \quad (3.24)$$

This shows that

$$\mathcal{B}_l\,\mathcal{B}_m R_j = a_{ml} R_j + b_{ml}(1-n)\, y_j\,. \quad (3.25)$$

If and only if

$$\mathcal{B}_l\,\mathcal{B}_m\big(R_j -\, y_j\,R\,\big) = a_{ml}\big(R_j -\, y_j\,R\,\big)\,. \quad (3.26)$$

Transvecting (3.4) and (3.21) by g^{jk}, using (1.4c), (1.4d), (1.12e) and (1.13h), we get

$$\mathcal{B}_l\,\mathcal{B}_m R^i_h = a_{ml} R^i_h - 2\mu_m \mathcal{B}_q y^q\big(C^i_{hl} - \delta^i_h C_l\,\big) - \tfrac{1}{2}\,\mathcal{B}_q y^q\big(W^i_k C^k_{hl} - W^i_h C_l\,\big)$$

$$-\tfrac{2}{3}\mathcal{B}_q\,y^q\,C_l\,\mathcal{B}_m R^i_h - \tfrac{2}{3}\mathcal{B}_q\,y^q \mathcal{B}_l\big(C_m R^i_h\,\big)\,. \quad (3.27)$$

This shows that

$$\mathcal{B}_l\,\mathcal{B}_m R^i_h = a_{ml} R^i_h\,. \quad (3.28)$$

If and only if

$$2\mu_m \mathcal{B}_q y^q\big(C^i_{hl} - \delta^i_h C_l\big) + \tfrac{1}{2}\,\mathcal{B}_q y^q\big(W^i_k C^k_{hl} - W^i_h C_l\,\big) + \tfrac{2}{3}\mathcal{B}_q\,y^q C_l\,\mathcal{B}_m R^i_h$$

$$+\tfrac{2}{3}\mathcal{B}_q\,y^q \mathcal{B}_l\big(C_m R^i_h\,\big) = 0\;\;. \quad (3.29)$$

And

$$\mathcal{B}_l\,\mathcal{B}_m R = a_{ml} R\; + b_{ml}\,(1-n) + \tfrac{1}{4}\mathcal{B}_l W + \tfrac{1}{4}\lambda_m\,W - 2\mu_m \mathcal{B}_q y^q\,(1-n)\,\mathrm{C}_l$$

$$-\tfrac{1}{2}\,\mathcal{B}_q\,y^q W^i_k\,C^k_{hl} - \tfrac{2}{3}\mathcal{B}_q\,y^q\,C_l\,\mathcal{B}_m R - \tfrac{2}{3}\mathcal{B}_q\,y^q \mathcal{B}_l(C_m R\,)\;\;. \quad (3.30)$$

This shows that

$$\mathcal{B}_l\,\mathcal{B}_m R\; = a_{ml} R\; + b_{ml}(1-n)\,. \quad (3.31)$$

If and only if

$$\tfrac{1}{4}\mathcal{B}_l W + \tfrac{1}{4}\lambda_m W - 2\mu_m \mathcal{B}_q y^q\,(1-n)\mathrm{C}_l - \tfrac{1}{2}\,\mathcal{B}_q y^q W^i_k\,C^k_{hl} - \tfrac{2}{3}\mathcal{B}_q\,y^q\,C_l\,\mathcal{B}_m R$$

$$-\tfrac{2}{3}\mathcal{B}_q\,y^q \mathcal{B}_l(C_m R\,) = 0\;\;\;. \quad (3.32)$$

In conclusion the proof of theorem is completed, we get

Theorem 3.6. In $G\mathcal{B}W\text{-}BRF_n$, vector R_j, the projective deviation tensor R^i_h and scalar R are given in (3.25), (3.28) and (3.31) if and only if the conditions (3.26), (3.29) and (3.32) are holds good, respectively.

Transvecting (3.4) by g_{ir}, using (1.1d), (1.12d) and (1.13f), we get

$$\mathcal{B}_l\,\mathcal{B}_m R_{rjkh} = a_{ml} R_{rjkh} + b_{ml}\big(g_{rk}\;g_{jh} - g_{rh}\,g_{jk}\big) + \tfrac{1}{3} a_{ml}\big(g_{rk}\,R_{jh} - g_{jk}\,R_{rh}\big)$$

$$+\tfrac{1}{4}\,\big((\mathcal{B}_l W_{rk})\,g_{jh}\; - (\mathcal{B}_l W_{rh}) g_{jk}\,\big) + \tfrac{1}{4}\lambda_m\big(W_{rk}\,g_{jh} - W_{rh}\,g_{jk}\,\big)$$

$$-2\mu_m \mathcal{B}_q y^q\big(\,g_{rk}\;C_{jhl}\; - g_{rh}\,C_{jkl}\,\big) - \tfrac{1}{2}\,\mathcal{B}_q y^q\big(W_{rk} C_{jhl}\; - W_{rh} C_{jkl}\,\big)$$

$$-\tfrac{1}{3}\mathcal{B}_l\,\mathcal{B}_m\big(g_{rk}\,R_{jh}-g_{jk}\,R_{rh}\big)-\tfrac{2}{3}\mathcal{B}_q\,y^q\,C_{jkl}\mathcal{B}_m R_{rh}-\tfrac{2}{3}\mathcal{B}_q\,y^q\mathcal{B}_l\big(C_{jkm}R_{rh}\big)\,. \tag{3.33}$$

This shows that

$$\mathcal{B}_l\mathcal{B}_m R_{rjkh}=a_{ml}R_{rjkh}+b_{ml}\big(g_{rk}\,g_{jh}-g_{rh}\,g_{jk}\big)+\tfrac{1}{4}\big((\mathcal{B}_l W_{rk})\,g_{jh}-(\mathcal{B}_l W_{rh})g_{jk}\big)$$

$$+\tfrac{1}{4}\lambda_m\,\big(W_{rk}\,g_{jh}-W_{rh}\,g_{jk}\big)-2\mu_m\mathcal{B}_q y^q\big(\,g_{rk}\;C_{jhl}-g_{rh}\,C_{jkl}\big)$$

$$-\tfrac{1}{2}\,\mathcal{B}_q y^q\big(W_{rk}\,C_{jhl}-W_{rh}C_{jkl}\big)-\tfrac{2}{3}\mathcal{B}_q\,y^q C_{jkl}\,\mathcal{B}_m R_{rh}-\tfrac{2}{3}\mathcal{B}_q\,y^q\mathcal{B}_l\big(C_{jkm}R_{rh}\big)\,. \tag{3.34}$$

If and only if

$$\mathcal{B}_l\,\mathcal{B}_m\big(\,g_{rk}\,R_{jh}-g_{jk}\,R_{rh}\big)=a_{ml}\big(\,g_{rk}\,R_{jh}-g_{jk}\,R_{rh}\big)\ . \tag{3.35}$$

Thus, the proof of theorem is completed, we get

Theorem 3.7. In GBW-BRF$_n$, associate tensor R_{rjkh} (Cartan's 3th curvature tensor R^i_{jkh}) is a generalized birecurrent Finsler space if and only if the condition (3.35) holds good.

It is known that Cartan's 3th curvature tensor R^i_{jkh} and Cartan's 4th curvature tensor K^i_{jkh} are connected by the formula [1]

$$R^i_{jkh}=K^i_{jkh}+C^i_{jp}\,H^p_{kh}\ . \tag{3.36}$$

Taking the covariant derivative of (3.36), with respect to x^m and x^l in the sence of Berwald we get

$$\mathcal{B}_l\mathcal{B}_m\,R^i_{jkh}=\mathcal{B}_l\,\mathcal{B}_m K^i_{jkh}+\mathcal{B}_l\,\mathcal{B}_m(C^i_{jp}\,H^p_{kh})\,. \tag{3.37}$$

Using (3.4) and (3.36) in (3.37) we get

$$\mathcal{B}_l\,\mathcal{B}_m\,K^i_{jkh}=a_{ml}K^i_{jkh}+b_{ml}\big(\delta^i_k\,g_{jh}-\delta^i_h g_{jk}\big)-\tfrac{1}{3}\mathcal{B}_l\,\mathcal{B}_m\big(\delta^i_k\,R_{jh}-g_{jk}\,R^i_h\big)$$

$$+\tfrac{1}{3}a_{ml}\big(\,\delta^i_k\,R_{jh}-g_{jk}\,R^i_h\big)-\mathcal{B}_l\,\mathcal{B}_m\,\big(C^i_{jp}\,H^p_{kh}\big)+a_{ml}\,C^i_{jp}\,H^p_{kh}$$

$$+\tfrac{1}{4}\big((\mathcal{B}_l W^i_k)g_{jh}-(\mathcal{B}_l W^i_h)g_{jk}\big)+\tfrac{1}{4}\lambda_m\big(W^i_k g_{jh}-W^i_h g_{jk}\big)-2\mu_m\mathcal{B}_q y^q\big(\delta^i_k C_{jhl}-\delta^i_h C_{jkl}\big)$$

$$-\tfrac{1}{2}\,\mathcal{B}_q y^q\big(W^i_k\,C_{jhl}-W^i_h C_{jkl}\big)-\tfrac{2}{3}\mathcal{B}_q\,y^q\,C_{jkl}\,\mathcal{B}_m R^i_h-\tfrac{2}{3}\mathcal{B}_q\,y^q\mathcal{B}_l\big(C_{jkm}R^i_h\big)\ . \tag{3.38}$$

This shows that

$$\mathcal{B}_l\,\mathcal{B}_m K^i_{jkh}=a_{ml}K^i_{jkh}+b_{ml}\,\big(\delta^i_k\,g_{jh}-\delta^i_h g_{jk}\big)+\tfrac{1}{4}\,\big((\mathcal{B}_l W^i_k)\,g_{jh}-(\mathcal{B}_l W^i_h)g_{jk}\big)$$

$$+\tfrac{1}{4}\lambda_m\big(W^i_k g_{jh}-W^i_h g_{jk}\big)-2\mu_m\mathcal{B}_q y^q\big(\delta^i_k C_{jhl}-\delta^i_h C_{jkl}\big)-\tfrac{1}{2}\,\mathcal{B}_q y^q\big(W^i_k C_{jhl}-W^i_h C_{jkl}\big)$$

$$-\tfrac{2}{3}\mathcal{B}_q\,y^q\,C_{jkl}\,\mathcal{B}_m R^i_h-\tfrac{2}{3}\mathcal{B}_q\,y^q\mathcal{B}_l\big(C_{jkm}R^i_h\big)\ . \tag{3.39}$$

If and only if

$$\mathcal{B}_l\,\mathcal{B}_m\big(\,\delta^i_k\,R_{jh}-\,g_{jk}\,R^i_h\big)=a_{ml}\big(\,\delta^i_k\,R_{jh}-\,g_{jk}\,R^i_h\big)\,. \tag{3.40}$$

and

$$\mathcal{B}_l\,\mathcal{B}_m\,\big(C^i_{jp}\,H^p_{kh}\big)=a_{ml}C^i_{jp}H^p_{kh}\ . \tag{3.41}$$

Thus, the proof of theorem is completed, we get

Theorem 3.8. In G$\mathcal{B}$W-BRF$_n$, Cartan's 4th curvature tensor K^i_{jkh} is a generalized birecurrent Finsler space if and only if the tensors $\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right)$ and $\left(C^i_{jp} H^p_{kh}\right)$ are a generalized birecurrent Finsler space.

Contracting the indices i and h in the equations (3.38) and using (1.1d), (1.1b), (1.13i), (1.13e), (1.14e), (1.12d) and (1.12b), we get

$$\mathcal{B}_l \mathcal{B}_m K_{jk} = a_{ml} K_{jk} + b_{ml}(1-n)\, g_{jk} + \tfrac{1}{4}\mathcal{B}_l W_{jk} + \tfrac{1}{4}\lambda_m W_{jk} - 2\mu_m \mathcal{B}_q y^q (1-n) C_{jkl}$$
$$- \tfrac{1}{2}\mathcal{B}_q y^q W^i_k C_{jil} - \tfrac{1}{3}\mathcal{B}_l \mathcal{B}_m\left(R_{jk} - g_{jk} R\right) + \tfrac{1}{3} a_{ml}\left(R_{jk} - g_{jk} R\right)$$
$$-\tfrac{2}{3}\mathcal{B}_q y^q C_{jkl} \mathcal{B}_m R - \tfrac{2}{3}\mathcal{B}_q y^q \mathcal{B}_l\left(C_{jkm} R\right) - \mathcal{B}_l \mathcal{B}_m\left(C^i_{jp} H^p_{ki}\right) + a_{ml} C^i_{jp} H^p_{ki} . \quad (3.42)$$

This shows that

$$\mathcal{B}_l \mathcal{B}_m K_{jk} = a_{ml} K_{jk} + b_{ml}(1-n)\, g_{jk} + \tfrac{1}{4}\mathcal{B}_l W_{jk} + \tfrac{1}{4}\lambda_m W_{jk} - 2\mu_m \mathcal{B}_q y^q (1-n) C_{jkl}$$
$$-\tfrac{1}{2}\mathcal{B}_q y^q W^i_k C_{jil} - \tfrac{2}{3}\mathcal{B}_q y^q C_{jkl} \mathcal{B}_m R - \tfrac{2}{3}\mathcal{B}_q y^q \mathcal{B}_l\left(C_{jkm} R\right). \quad (3.43)$$

If and only if

$$\mathcal{B}_l \mathcal{B}_m\left(R_{jk} - g_{jk} R\right) = a_{ml}\left(R_{jk} - g_{jk} R\right). \quad (3.44)$$

and

$$\mathcal{B}_l \mathcal{B}_m\left(C^i_{jp} H^p_{kh}\right) = a_{ml} C^i_{jp} H^p_{kh} . \quad (3.45)$$

Transvecting (3.42) by y^k, using (1.6b), (1.1a), (1.1c), (1.4b), (1.12a), (1.12f), (1.14c) and (1.13d), we get

$$\mathcal{B}_l \mathcal{B}_m K_j = a_{ml} K_j + b_{ml}(1-n)\, y_j + \tfrac{1}{3} a_{ml}\left(R_j - y_j R\right) - \tfrac{1}{3}\mathcal{B}_l \mathcal{B}_m\left(R_j - y_j R\right)$$
$$-\mathcal{B}_l \mathcal{B}_m\left(C^i_{jp} H^p_i\right) + a_{ml} C^i_{jp} H^p_i . \quad (3.46)$$

This shows that

$$\mathcal{B}_l \mathcal{B}_m K_j = a_{ml} K_j + b_{ml}(1-n)\, y_j . \quad (3.47)$$

If and only if

$$\mathcal{B}_l \mathcal{B}_m\left(R_j - y_j R\right) = a_{ml}\left(R_j - y_j R\right). \quad (3.48)$$

and

$$\mathcal{B}_l \mathcal{B}_m\left(C^i_{jp} H^p_i\right) = a_{ml} C^i_{jp} H^p_i. \quad (3.49)$$

Thus, the proof of theorem is completed, we get

Theorem 3.9. In G$\mathcal{B}$W-BRF$_n$, K−Ricci tensor K_{jk} and curvature vector K_j is given in (3.43) and (3.47), if and only if the conditions (3.44), (3.45), (3.48) and (3.49) are holds good, respectively.

Transvecting (3.38) and (3.42) by g^{jk},respectively using (1.4c), (1.4d), (1.12e), (1.14d), (1.14f) and (1.13h), we get

$$\mathcal{B}_l\,\mathcal{B}_m K^i_h = a_{ml}K^i_h - g^{jk}\mathcal{B}_l\,\mathcal{B}_m\left(C^i_{jp}\,H^p_{kh}\right) + a_{ml}g^{jk}C^i_{jp}\,H^p_{kh} - 2\mu_m\mathcal{B}_q y^q\left(C^i_{hl} - \delta^i_h C_l\right)$$
$$-\tfrac{1}{2}\,\mathcal{B}_q y^q\left(W^i_k C^k_{hl} - W^i_h C_l\right) - \tfrac{2}{3}\mathcal{B}_q\,y^q\,C_l\,\mathcal{B}_m R^i_h - \tfrac{2}{3}\mathcal{B}_q\,y^q\mathcal{B}_l\left(C_m R^i_h\right). \qquad (3.50)$$

This shows that

$$\mathcal{B}_l\,\mathcal{B}_m K^i_h = a_{ml}K^i_h - 2\mu_m\mathcal{B}_q y^q\left(C^i_{hl} - \delta^i_h C_l\right) - \tfrac{1}{2}\,\mathcal{B}_q y^q\left(W^i_k C^k_{hl} - W^i_h C_l\right)$$
$$-\tfrac{2}{3}\mathcal{B}_q\,y^q\,C_l\,\mathcal{B}_m R^i_h - \tfrac{2}{3}\mathcal{B}_q\,y^q\mathcal{B}_l\left(C_m R^i_h\right). \qquad (3.51)$$

If and only if

$$\mathcal{B}_l\,\mathcal{B}_m\left(C^i_{jp}\,H^p_{kh}\right) = a_{ml}\,C^i_{jp}\,H^p_{kh}\ .\ \text{ Where }\ g^{jk} \neq 0\ . \qquad (3.52)$$

And

$$\mathcal{B}_l\,\mathcal{B}_m K = a_{ml}K + b_{ml}\,(1-n) - g^{jk}\mathcal{B}_l\,\mathcal{B}_m\left(C^i_{jp}\,H^p_{kh}\right) + a_{ml}\,g^{jk}C^i_{jp}\,H^p_{kh} + \tfrac{1}{4}\,\mathcal{B}_l W$$
$$+\tfrac{1}{4}\lambda_m W - 2\mu_m\mathcal{B}_q y^q\,(1-n)C_l - \tfrac{1}{2}\,\mathcal{B}_q y^q W^i_k\,C^k_{hl} - \tfrac{2}{3}\mathcal{B}_q\,y^q\,C_l\,\mathcal{B}_m R$$
$$-\tfrac{2}{3}\mathcal{B}_q\,y^q\mathcal{B}_l(C_m R)\ . \qquad (3.53)$$

This shows that

$$\mathcal{B}_l\,\mathcal{B}_m K = a_{ml}K + b_{ml}(1-n) + \tfrac{1}{4}\mathcal{B}_l W + \tfrac{1}{4}\lambda_m\,W - 2\mu_m\mathcal{B}_q y^q\,(1-n)\,C_l$$
$$-\tfrac{1}{2}\,\mathcal{B}_q\,y^q W^i_k\,C^k_{hl} - \tfrac{2}{3}\mathcal{B}_q\,y^q C_l\,\mathcal{B}_m R - \tfrac{2}{3}\mathcal{B}_q\,y^q\mathcal{B}_l(C_m R)\,. \qquad (3.54)$$

If and only if

$$\mathcal{B}_l\,\mathcal{B}_m\left(C^i_{jp}\,H^p_{kh}\right) = a_{ml}\,C^i_{jp}\,H^p_{kh}\ .\text{ Where }\ g^{jk} \neq 0\ . \qquad (3.55)$$

Thus, we conclude

Theorem 3.10. In G$\mathcal{B}$W-BRF$_n$, the projective deviation tensor R^i_h and scalar R are given in (3.51) and (3.54) if and only if the conditions (3.52) and (3.55) are holds good, respectively.

CONCLUSIONS

In this paper, we have presented a detailed study of the decomposition of curvature tensors in Finsler spaces using higher-order derivatives of Berwald and Cartan connections. Our analysis has revealed new properties and relationships between the components of the decomposed tensors. These findings contribute to a better understanding of the geometric structure of Finsler spaces and may have implications for various applications.

A generalized $\mathcal{B}$W-birecurrent space in Finsler space is satisfied in condition (2.3).

In $G\mathcal{B}W\text{-}BRF_n$, $\mathcal{B}$-covariant derivatives of the second orders for torsion tensor W^i_{kh} and deviation tensor W^i_h are given by (2.4) and (2.5).
In $G\mathcal{B}W\text{-}FRF_n$, the condition of being necessary and sufficient for the Cartan's 3th tensor R^i_{jkh} is a generalized birecurrent if the equation (3.6) holds. In $G\mathcal{B}W\text{-}BRF_n$, Ricci tensor R_{jk} is a generalized birecurrent if the equation (3.23) holds. In $G\mathcal{B}W\text{-}BRF_n$, the associate curvature tensor R_{jrkh} is a generalized birecurrent if the condition (3.35) holds good.
In $G\mathcal{B}W\text{-}BRF_n$,we get the same relationship between the Weyls projective curvature tensor W^i_{jkh} and the tensors R^i_{jkh} and K^i_{jkh}.

In $G\mathcal{B}W\text{-}BRF_n$, the Cartan's 4th tensor K^i_{jkh} is a generalized birecurrent if and only if the tensors $\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right)$ and $\left(C^i_{jp} H^p_{kh}\right)$ are a generalized birecurrent Finsler space.
In $G\mathcal{B}W\text{-}BRF_n$, $\mathcal{B}$-covariant derivatives of the second orders for K-Ricci tensor K_{jk} is a generalized birecurrent if and only if the tensors $\left(R_{jk} - g_{jk} R\right)$ and $\left(C^i_{jp} H^p_{ki}\right)$ are a generalized birecurrent Finsler space.
The authors argue that further study and advancement in a generalized $\mathcal{B}$W-birecurrent Finsler spaces is necessary and tie it in with Finsler space's distinctive space features.

Recommendations

Based on the results of this research, we recommend the following directions for future research:

- Explore other types of decompositions: Investigate different decomposition schemes and their corresponding geometric interpretations.
- Investigate the physical implications: Explore the physical implications of the decomposition results, particularly in the context of field theories and cosmology.
- Develop numerical methods: Develop numerical methods for computing the decomposed tensors and analyzing their properties.

Disclaimer

The article has not been previously presented or published, and is not part of a thesis project.

Conflict of Interest

There are no financial, personal, or professional conflicts of interest to declare.

Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript**.**

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